Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 153 (2006) 3220–3229
Topology and its Applications
www.elsevier.com/locate/topol

Characterizations of normal covers of rectangular products

Yukinobu Yajima

Department of Mathematics, Kanagawa University, Yokohama 221-8686, Japan

Received 6 September 2004; received in revised form 11 January 2005; accepted 11 January 2005

Abstract

Stone, Michael and Morita have given various equivalent conditions for normal covers of topological spaces. Here, as an analogue of the classic characterization, we give some characterizations for normal covers of rectangular products in terms of cozero rectangles. Moreover, we apply our characterizations to consider the base-paracompactness of rectangular products.

MSC: 54B10; 54D20

Keywords: Rectangular product; Cozero rectangular; Zero rectangular; Shrinking; X-rectangular; Strongly rectangular; Base-paracompact; σ-space; **DC**-like

1. Introduction

An open cover $\mathcal{O}$ of a topological space X is *normal* if there is a sequence $\{\mathcal{U}_n\}$ of open covers of X such that $\mathcal{U}_{n+1}$ is a star-refinement of $\mathcal{U}_n$ for each $n \in \omega$, where $\mathcal{U}_0 = \mathcal{O}$.

We may well know the following characterization of normal covers of topological spaces. For example, it is seen in [1, p. 122], [5, Theorems 1.2 and 1.4] and [7, Theorem 1.2] etc.

Theorem 1.1 *(Stone–Michael–Morita). Let X be a topological space and $\mathcal{O}$ an open cover of X. Then the following are equivalent*:

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero refinement which has a shrinking consisting of zero-sets.*

E-mail address: yajimy01@kanagawa-u.ac.jp (Y. Yajima).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2005.01.041

On the other hand, Pasynkov [14] introduced the concept of rectangular products in order to prove the following inequalities in dimension theory

$$\dim(X \times Y) \leqslant \dim X + \dim Y \quad \text{and} \quad \operatorname{Ind}(X \times Y) \leqslant \operatorname{Ind} X + \operatorname{Ind} Y.$$

In this paper, we give some characterizations analogous to the Stone–Michael–Morita's Theorem above for normal covers of rectangular products. Moreover, we can apply these characterizations to the base-paracompactness of rectangular products as well as in [21].

Throughout this paper, all spaces are *topological spaces* without any separation axiom. However, paracompact spaces are assumed to be *Hausdorff*.

2. Structural lemmas

Let X be a space and $\mathcal{U}$ a cover of X. A cover $\mathcal{V}$ of X is called a *refinement* of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. A cover $\{W_U\colon U \in \mathcal{U}\}$ of X is called a *shrinking* of $\mathcal{U}$ if $\overline{W_U} \subset U$ for each $U \in \mathcal{U}$. Let $\mathcal{A}$ be a collection of subsets in X and let $\mathcal{B} = \{B_A\colon A \in \mathcal{A}\}$. We say that $\mathcal{B}$ is a *partial shrinking* of $\mathcal{A}$ if $\overline{B_A} \subset A$ for each $A \in \mathcal{A}$. We say that $\mathcal{B}$ is a *swelling* of $\mathcal{A}$ if $A \subset B_A$ for each $A \in \mathcal{A}$.

Let $X \times Y$ be a product space. A subset of the form $A \times B$ in $X \times Y$ is called a *rectangle*. For a subset R in $X \times Y$, R' and R'' denote the projections of R into X and Y, respectively. It is clear that $R = R' \times R''$ iff R is a rectangle in $X \times Y$. A rectangle R is called a *cozero rectangle* (*zero rectangle*) if R' and R'' are cozero-sets (zero-sets) in X and Y, respectively. Note that R is a cozero rectangle (zero rectangle) in $X \times Y$ iff it is a cozero-set (zero-set) and a rectangle in $X \times Y$. A cover $\mathcal{G}$ of $X \times Y$ is said to be *cozero rectangular* (respectively*, zero rectangular, rectangular*) if each member of $\mathcal{G}$ is a cozero rectangle (respectively, zero rectangle, rectangle) in $X \times Y$.

A product space $X \times Y$ is said to be *rectangular* [14] if every finite cozero cover (equivalently, every normal cover) of $X \times Y$ has a σ-locally finite cozero rectangular refinement (see [6, Lemma 1]).

Structural Lemma I. *Let $X \times Y$ be a product space and $\mathcal{O}$ a cozero rectangular cover of $X \times Y$. Suppose that there is a zero rectangular cover $\{F_\alpha \times K_\alpha\colon \alpha \in \Omega_n$ and $n \in \omega\}$ of $X \times Y$, satisfying*

(a) *$\{F_\alpha\colon \alpha \in \Omega_n\}$ is discrete in X and has a discrete swelling consisting of cozero-sets in X for each $n \in \omega$,*
(b) *for each $\alpha \in \Omega_n$, $n \in \omega$, there is some $\mathcal{O}_\alpha \subset \mathcal{O}$ in X such that*
  (i) *$O \in \mathcal{O}_\alpha$ implies $F_\alpha \subset O'$, and*
  (ii) *$\{O''\colon O \in \mathcal{O}_\alpha\}$ is locally finite in Y and covers K_α.*

Then $\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement $\mathcal{G}$ such that $\mathcal{G}$ has a zero rectangular shrinking.

Proof. The proof is obtained by a modification of that of [21, Lemma 3.2]. However, for the accuracy, we mainly restate the part of inductive construction.

Let $\mathcal{F}_n = \{F_\alpha\colon \alpha \in \Omega_n\}$ for each $n \in \omega$. By (a), for each $n \in \omega$, there are three discrete swellings $\{W_{\alpha,i}\colon \alpha \in \Omega_n\}$, $i \in 2$, and $\{L_\alpha\colon \alpha \in \Omega_n\}$ of $\mathcal{F}_n$ in X, satisfying for each $\alpha \in \Omega_n$,

(iii) $W_{\alpha,0}$ and $W_{\alpha,1}$ are cozero-sets in X,
(iv) L_α is a zero-set in X,
(v) $F_\alpha \subset W_{\alpha,0} \subset L_\alpha \subset W_{\alpha,1}$.

Now, for each $n \in \omega$, we will construct a collection $\mathcal{G}_n$ of cozero rectangles, two collections $\mathcal{S}_n$ and $\mathcal{Z}_n$ of zero rectangles, a function $\varphi^n_{n-1}\colon \mathcal{Z}_n \to \mathcal{Z}_{n-1}$ and a sequence $\{H_k(Z)\colon Z \in \mathcal{Z}_n\}$, $k \in \omega$, of swellings of $\mathcal{Z}_n$ consisting of cozero rectangles, satisfying the following conditions:

(1) $\mathcal{G}_n$ is locally finite and σ-discrete in $X \times Y$.
(2) $\mathcal{S}_n$ is a partial shrinking of $\mathcal{G}_n$.
(3) $\mathcal{Z}_n$ is locally finite and σ-discrete in $X \times Y$.
(4) $\{H_0(Z)\colon Z \in \mathcal{Z}_n\}$ is locally finite and σ-discrete in $X \times Y$.

(5) For each $Z \in \mathcal{Z}_n$, $Z \subset \varphi^n_{n-1}(Z)$.
(6) For each $Z \in \mathcal{Z}_n$, $Z = \bigcap_{k\in\omega} H_k(Z)$ and $\overline{H_{k+1}(Z)} \subset H_k(Z) \subset H_k(\varphi^n_{n-1}(Z))$ $(k = 0, 1, \ldots)$.
(7) $(\bigcup_{i\leqslant n} \mathcal{S}_i) \cup \mathcal{Z}_n$ covers $X \times Y$.
(8) Each member of $\mathcal{G}_n$ is contained in some member of $\mathcal{O}$.
(9) $\bigcup \mathcal{G}_n \subset \bigcup\{H_{n-1}(Z)\colon Z \in \mathcal{Z}_{n-1}\}$.
(10) For each $Z \in \mathcal{Z}_n$, Z' meets at most one member of $\mathcal{F}_n$.
(11) For each $Z \in \mathcal{Z}_n$ and for each $\alpha \in \Omega_n$, $Z' \cap F_\alpha \neq \emptyset$ implies $Z'' \cap K_\alpha = \emptyset$.

Assume that we have already constructed the above ones for $i \leqslant n$.

Fix a $Z \in \mathcal{Z}_n$. First, assume that Z' meets $\bigcup \mathcal{F}_{n+1}$. Take any $\alpha \in \Omega_{n+1}$ with $F_\alpha \cap Z' \neq \emptyset$. Let $\mathcal{O}_\alpha = \{U_\lambda \times V_\lambda\colon \lambda \in \Lambda_\alpha\}$.

Let $\Lambda_\alpha(Z) = \{\lambda \in \Lambda_\alpha\colon (U_\lambda \times V_\lambda) \cap (F_\alpha \times K_\alpha) \cap Z \neq \emptyset\}$. For each $\lambda \in \Lambda_\alpha(Z)$, by (i) in (b), there are a cozero-set $B_{\lambda,\alpha}$ and a zero-set $C_{\lambda,\alpha}$ in X such that

$$F_\alpha \cap Z' \subset B_{\lambda,\alpha} \subset C_{\lambda,\alpha} \subset U_\lambda \cap W_{\alpha,0} \cap H_n(Z)'.$$

By (ii) in (b), $\{V_\lambda\colon \lambda \in \Lambda_\alpha(Z)\} \cup \{Y \setminus (K_\alpha \cap Z'')\}$ is a normal cover of Y. So we may assume from Theorem 1.1 that $\{V_\lambda\colon \lambda \in \Lambda_\alpha(Z)\}$ is not only locally finite but also σ-discrete in Y. Since $\bigcup\{V_\lambda\colon \lambda \in \Lambda_\alpha(Z)\}$ is a cozero-set containing a zero-set $K_\alpha \cap Z''$ in Y, there are two zero-sets $M(Z,\alpha)$ and $N(Z,\alpha)$ in Y such that $M(Z,\alpha) \cup N(Z,\alpha) = Z''$ and

(viii) $K_\alpha \cap Z'' \subset Z'' \setminus N(Z,\alpha) \subset M(Z,\alpha) \subset (\bigcup\{V_\lambda\colon \lambda \in \Lambda_\alpha(Z)\}) \cap Z''$.

Moreover, there is a locally finite and σ-discrete collection $\{D_\lambda\colon \lambda \in \Lambda_\alpha(Z)\}$ of zero-sets in Y such that

(vi) $D_\lambda \subset V_\lambda \cap Z''$ for each $\lambda \in \Lambda_\alpha(Z)$, and
(vii) $\bigcup\{D_\lambda\colon \lambda \in \Lambda_\alpha(Z)\} = M(Z,\alpha)$.

Here, for each $\lambda \in \Lambda_\alpha(Z)$, let

$$Z_\lambda(\alpha) = \big((L_\alpha \setminus B_{\lambda,\alpha}) \cap Z'\big) \times D_\lambda \quad \text{and}$$
$$H_0\big(Z_\lambda(\alpha)\big) = (W_{\alpha,1} \times V_\lambda) \cap H_0(Z).$$

Let $Z(\alpha) = (L_\alpha \cap Z') \times N(Z,\alpha)$ and $H_0(Z(\alpha)) = (W_{\alpha,1} \times Y) \cap H_0(Z)$. Moreover, let $Z^* = (Z' \setminus \bigcup\{W_{\beta,0}\colon \beta \in \Omega_{n+1}\}) \times Z''$ and $H_0(Z^*) = H_0(Z)$.

Now, for the $Z \in \mathcal{Z}_n$ meeting $\bigcup \mathcal{F}_{n+1}$, we put

$$\mathcal{Z}_{n+1}(Z) = \big\{Z_\lambda(\alpha)\colon \lambda \in \Lambda_\alpha(Z) \text{ and } \alpha \in \Omega_{n+1} \text{ with } F_\alpha \cap Z' \neq \emptyset\big\}$$
$$\cup \big\{Z(\alpha)\colon \alpha \in \Omega_{n+1} \text{ with } F_\alpha \cap Z' \neq \emptyset\big\} \cup \{Z^*\},$$
$$\mathcal{G}_{n+1}(Z) = \big\{\big((U_\lambda \cap W_{\alpha,0}) \times V_\lambda\big) \cap H_n(Z)\colon \lambda \in \Lambda_\alpha(Z) \text{ and } \alpha \in \Omega_{n+1} \text{ with } F_\alpha \cap Z' \neq \emptyset\big\}, \quad \text{and}$$
$$\mathcal{S}_{n+1}(Z) = \big\{C_{\lambda,\alpha} \times D_\lambda\colon \lambda \in \Lambda_\alpha(Z) \text{ and } \alpha \in \Omega_{n+1} \text{ with } F_\alpha \cap Z' \neq \emptyset\big\}.$$

For each $Z \in \mathcal{Z}_n$ disjoint from $\bigcup \mathcal{F}_{n+1}$, we put $\mathcal{Z}_{n+1}(Z) = \{Z\}$ and $\mathcal{G}_{n+1}(Z) = \mathcal{S}_{n+1}(Z) = \{\emptyset\}$. Here, letting Z range over $\mathcal{Z}_n$, we put

$$\mathcal{Z}_{n+1} = \bigcup\big\{\mathcal{Z}_{n+1}(Z)\colon Z \in \mathcal{Z}_n\big\}, \quad \mathcal{G}_{n+1} = \bigcup\big\{\mathcal{G}_{n+1}(Z)\colon Z \in \mathcal{Z}_n\big\} \quad \text{and}$$
$$\mathcal{S}_{n+1} = \bigcup\big\{\mathcal{S}_{n+1}(Z)\colon Z \in \mathcal{Z}_n\big\}.$$

The function $\varphi^{n+1}_n : \mathcal{Z}_{n+1} \to \mathcal{Z}_n$ is defined by $\varphi^{n+1}_n(\mathcal{Z}_{n+1}(Z)) = \{Z\}$ for each $Z \in \mathcal{Z}_n$.

For each $\widehat{Z} \in \mathcal{Z}_{n+1}$ with $\varphi^{n+1}_n(\widehat{Z}) = Z$, that is, $\widehat{Z} \in \mathcal{Z}_{n+1}(Z)$, we can choose a sequence $\{H_k(\widehat{Z})\}$ of cozero rectangles such that $\widehat{Z} = \bigcap_{k\in\omega} H_k(\widehat{Z})$ for each $\widehat{Z} \in \mathcal{Z}_{n+1}$ and $\overline{H_{k+1}(\widehat{Z})} \subset H_k(\widehat{Z}) \subset H_k(Z)$ for each $k \in \omega$. Thus, we have constructed the desired $\mathcal{G}_{n+1}, \mathcal{S}_{n+1}, \langle \mathcal{Z}_{n+1}, \varphi^{n+1}_n\rangle$ and $\{H_k(\widehat{Z})\colon \widehat{Z} \in \mathcal{Z}_{n+1}\}_{k\in\omega}$. Then all the conditions (1)–(11) are satisfied. We have completed the induction.

Let $H_{n,k} = \bigcup\{H_k(Z)\colon Z \in \mathcal{Z}_n\}$ for each $n, k \in \omega$. Then we can establish the same claims as in the proof of [21, Lemma 3.2]:

Claim 1. $\bigcap_{n\in\omega}(\bigcup \mathcal{Z}_n) = \emptyset$.

Claim 2. $\bigcup \mathcal{Z}_n = \bigcap_{k\in\omega} H_{n,k} = \bigcap_{k\in\omega} \overline{H}_{n,k}$ *for each* $n \in \omega$.

Let $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$ and let $\mathcal{S} = \bigcup_{n\in\omega} \mathcal{S}_n$. It follows from Claim 1 and (7) that $\mathcal{S}$ covers $X \times Y$. Hence, by (1), $\mathcal{G}$ is a σ-discrete cover $X \times Y$. By (2), $\mathcal{S}$ is a zero rectangular shrinking of $\mathcal{G}$. By (8), $\mathcal{G}$ is a cozero rectangular refinement of $\mathcal{O}$. It is shown in the same way as in the last part of the proof of [21, Lemma 3.2] that $\mathcal{G}$ is locally finite in $X \times Y$. Hence $\mathcal{G}$ and $\mathcal{S}$ are the desired ones. □

Under the assumption of paracompactness of X and Y, the conditions of the above lemma become somewhat simpler as follows.

Structural Lemma II. *Let X and Y be paracompact spaces. Let $\mathcal{O}$ be an open cover of $X \times Y$. Suppose that there is a closed rectangular cover $\{F_\alpha \times K_\alpha\colon \alpha \in \Omega\}$ of $X \times Y$, satisfying*

(c) *$\{F_\alpha\colon \alpha \in \Omega\}$ is a σ-locally finite closed cover of X,*
(d) *for each $\alpha \in \Omega$, there is a collection $\mathcal{V}_\alpha$ of open sets in Y such that*
  (i) *$\mathcal{V}_\alpha$ covers K_α, and*
  (ii) *for each $V \in \mathcal{V}_\alpha$, there is a finite collection $\mathcal{U}_V$ of open sets in X such that $F_\alpha \subset \bigcup \mathcal{U}_V$ and $U \times V$ is contained in some member of $\mathcal{O}$ for each $U \in \mathcal{U}_V$.*

Then $\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.

Proof. By (c), we can let $\Omega = \bigcup_{n\in\omega} \Omega_n$ such that $\{F_\alpha\colon \alpha \in \Omega_n\}$ is locally finite in X for each $n \in \omega$. Take an $n \in \omega$. Since X is subparacompact, note that there is a σ-discrete closed cover $\{E_\xi\colon \xi \in \Xi_n\}$ of X such that each E_ξ meets at most finitely many member of $\{F_\alpha\colon \alpha \in \Omega_n\}$. Let us consider

$$\mathcal{R} = \{(E_\xi \cap F_\alpha) \times K_\alpha\colon \xi \in \Xi_n,\ \alpha \in \Omega_n \text{ and } n \in \omega\}$$

instead of $\{F_\alpha \times K_\alpha\colon \alpha \in \Omega\}$. Then $\{E_\xi \cap F_\alpha\colon \xi \in \Xi_n,\ \alpha \in \Omega_n \text{ and } n \in \omega\}$ is a σ-discrete closed cover of X. It is clear that $\mathcal{R}$ covers $X \times Y$. Moreover, it is easily seen that the condition (d) is satisfied. Thus, without loss of generality, we can assume that $\{F_\alpha\colon \alpha \in \Omega\}$ is σ-discrete in X. Since X is paracompact, the condition (a) in Structural Lemma I is clearly satisfied. Moreover, by the normality of X and the paracompactness of Y, for each $\alpha \in \Omega$, it is easy to construct a collection $\{U_{\lambda,i} \times V_\lambda\colon i \leqslant k_\lambda \text{ and } \lambda \in \Lambda_\alpha\}$ of cozero rectangles such that

(iii) F_α is contained in U_λ for each $\lambda \in \Lambda_\alpha$, where $U_\lambda = \bigcup_{i\leqslant k_\lambda} U_{\lambda,i}$,
(iv) $\{V_\lambda\colon \lambda \in \Lambda_\alpha\}$ is locally finite and σ-discrete in Y and covers K_α,
(v) each $U_{\lambda,i} \times V_\lambda$ is contained in some member of $\mathcal{O}$.

It follows from Structural Lemma I that the cozero rectangular cover $\{U_\lambda \times V_\lambda\colon \alpha \in \Lambda_\alpha \text{ and } \alpha \in \Omega\}$ of $X \times Y$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking. Hence also $\mathcal{O}$ has this kind of a refinement. □

3. X-rectangular products

A product space $X \times Y$ is said to be *X-rectangular* [13] if every finite cozero cover $\mathcal{O}$ of $X \times Y$ has a cozero rectangular refinement $\mathcal{G}$ such that $\pi_X(\mathcal{G}) = \{G'\colon G \in \mathcal{G}\}$ is σ-locally finite in X (where $G, H \in \mathcal{G}$ with $G \neq H$ means that G' and H' are considered as different members even if they are coincided as a set).

Lemma 3.1. *A product space $X \times Y$ is X-rectangular if and only if every finite cozero cover $\mathcal{O}$ of $X \times Y$ has a cozero rectangular refinement $\mathcal{G}$ such that $\pi_X(\mathcal{G}) = \{G'\colon G \in \mathcal{G}\}$ is σ-discrete in X.*

Proof. We only show the "only if" part. Let $\mathcal{O}$ be a finite cozero cover of $X \times Y$. By the assumption, there is a cozero rectangular refinement $\{U_\lambda \times V_\lambda \colon \lambda \in \Lambda_n \text{ and } n \in \omega\}$ of $\mathcal{O}$ such that $\{U_\lambda \colon \lambda \in \Lambda_n\}$ is locally finite in X for each $n \in \omega$. Take an $n \in \omega$. For each $\lambda \in \Lambda_n$, there are two sequences $\{U_{\lambda,i}\}$ and $\{F_{\lambda,i}\}$ of cozero-sets and zero-sets, respectively, in X satisfying

(i) $U_{\lambda,i} \subset F_{\lambda,i} \subset U_\lambda$ for each $i \in \omega$, and
(ii) $U_\lambda = \bigcup_{i\in\omega} U_{\lambda,i} = \bigcup_{i\in\omega} F_{\lambda,i}$.

Take an $i \in \omega$. It follows from (i) and [5, Theorem 3.17] ([9, Proposition 4.2] or [12, Lemma 5.6]) that $\{U_{\lambda,i} \colon \lambda \in \Lambda_n\}$ is uniformly locally finite in X. That is, there is a σ-discrete cozero cover $\{W_\alpha \colon \alpha \in \Omega_{n,i}\}$ of X such that each W_α meets at most finitely many members of $\{U_{\lambda,i} \colon \lambda \in \Lambda_n\}$. Let

$$\mathcal{G}_{n,i} = \bigl\{(W_\alpha \cap U_{\lambda,i}) \times V_\lambda \colon \alpha \in \Omega_{n,i} \text{ and } \lambda \in \Lambda_n\bigr\}.$$

Note that $\pi_X(\mathcal{G}_{n,i}) = \{W_\alpha \cap U_{\lambda,i} \colon \alpha \in \Omega_{n,i} \text{ and } \lambda \in \Lambda_n\}$ is σ-discrete in X. Now, we let $\mathcal{G} = \bigcup\{\mathcal{G}_{n,i} \colon n, i \in \omega\}$. Then each member of $\mathcal{G}$ is a cozero rectangle in $X \times Y$ contained in some member of $\mathcal{O}$, and $\pi_X(\mathcal{G}) = \{G' \colon G \in \mathcal{G}\}$ is σ-discrete in X. It is easily seen that $\mathcal{G}$ is a cover of $X \times Y$. $\square$

Remark. X-rectangular products were originally defined for Tychonoff products in [13]. In the case of a Tychonoff product $X \times Y$, the proof of Lemma 3.1 can be obtained by a modification of that of [13, Theorem 2.2]. However, the assumption that Y is Tychonoff is necessary in his proof, because the Stone–Čech compactification βY of Y has to be used there.

Theorem 3.2. *Let $X \times Y$ be an X-rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent*:

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

Proof. (e) $\to$ (d) $\to$ (b) $\to$ (a) and (e) $\to$ (c) $\to$ (b): Obvious.

(a) $\to$ (e): Let $\{O_1, O_2\}$ be a binary cozero cover of $X \times Y$. By the assumption and Lemma 3.1, there is a cozero rectangular refinement $\{U_\lambda \times V_\lambda \colon \lambda \in \Lambda_n \text{ and } n \in \omega\}$ of $\{O_1, O_2\}$ such that $\{U_\lambda \colon \lambda \in \Lambda_n\}$ is discrete in X for each $n \in \omega$. Take an $n \in \omega$. For each $\lambda \in \Lambda_n$, there is a sequence $\{F_{\lambda,i} \times K_{\lambda,i}\}$ of zero rectangles such that $U_\lambda \times V_\lambda = \bigcup_{i\in\omega} F_{\lambda,i} \times K_{\lambda,i}$. Then we can see that $\{F_{\lambda,i} \times K_{\lambda,i} \colon \lambda \in \Lambda_n \text{ and } n, i \in \omega\}$ satisfies all the conditions of Structural Lemma I. Hence $\{O_1, O_2\}$ has a locally finite, σ-discrete, cozero rectangular refinement $\mathcal{G}$ which has a zero rectangular shrinking $\mathcal{Z}$.

Let $\mathcal{O} = \{O_\xi \colon \xi \in \Xi\}$ be a normal cover of $X \times Y$. We may assume that $\mathcal{O}$ is a locally finite, σ-discrete, cozero cover of $X \times Y$. There is a shrinking $\{S_\xi \colon \xi \in \Xi\}$ of $\mathcal{O}$ consisting of zero-sets. For each $\xi \in \Xi$, since $\{O_\xi, X \times Y \setminus S_\xi\}$ is a binary cozero cover of $X \times Y$, it follows from the above that there is a locally finite, σ-discrete, cozero rectangular refinement $\mathcal{G}_\xi$ which has a zero rectangular shrinking $\mathcal{Z}_\xi = \{Z_G \colon G \in \mathcal{G}_\xi\}$. Let $\mathcal{G}_\xi^+ = \{G \in \mathcal{G}_\xi \colon G \subset O_\xi\}$ and $\mathcal{Z}_\xi^+ = \{Z_G \colon G \in \mathcal{G}_\xi^+\}$ for each $\xi \in \Xi$. Then we have $S_\xi \subset \bigcup \mathcal{Z}_\xi^+ \subset \bigcup \mathcal{G}_\xi^+ \subset O_\xi$. Here, we let $\mathcal{G}^+ = \bigcup\{\mathcal{G}_\xi^+ \colon \xi \in \Xi\}$ and $\mathcal{Z}^+ = \bigcup\{\mathcal{Z}_\xi^+ \colon \xi \in \Xi\}$. It is easy to check that $\mathcal{G}^+$ and $\mathcal{Z}^+$ are the desired ones. $\square$

It is pointed out in the proof of [17, Theorem 1] that if X is a metric space, then the rectangularity of $X \times Y$ implies the X-rectangularity. So we have

Corollary 3.3. *Let X be a metric space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent*:

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

Remark. If a product space $X \times Y$ is not rectangular, there is a normal cover of $X \times Y$ which has no σ-locally finite cozero rectangular refinement. In fact, there is a non-rectangular product with a metric factor (see [16,17]). So we cannot exclude the assumption of rectangularity of $X \times Y$ in Corollary 3.3.

Consider the case that $\mathcal{O}$ is a countable normal cover which has a countable, cozero rectangular refinement. Then, applying Structural Lemma I directly, we can obtain the following without any assumption of $X \times Y$.

Proposition 3.4. *Let $\mathcal{O}$ be a normal countable cover of a product space $X \times Y$. Then the following are equivalent*:

(a) *$\mathcal{O}$ has a countable, cozero rectangular refinement.*
(b) *$\mathcal{O}$ has a locally finite, countable, cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a locally finite, countable, cozero rectangular refinement which has a zero rectangular shrinking.*

4. Products with a σ-space factor

Recall that a regular T_1-space X is a *σ-space* if there is a σ-discrete closed net of X.

Theorem 4.1. *Let X be a paracompact σ-space and Y a space. Let $\mathcal{O}$ be a normal cover of $X \times Y$. Then the following are equivalent.*

(a) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(b) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

Proof. We only show (a) $\to$ (d): Let $\mathcal{F} = \{F_\alpha\colon \alpha \in \Omega\}$ be a σ-discrete closed net of X, where each F_α is non-empty. We may assume that $\mathcal{O} = \{U_\lambda \times V_\lambda\colon \lambda \in \Lambda_n \text{ and } n \in \omega\}$ is a σ-locally finite cozero rectangular cover of $X \times Y$, where $\{U_\lambda \times V_\lambda\colon \lambda \in \Lambda_n\}$ is locally finite in $X \times Y$ for each $n \in \omega$. For each $\lambda \in \Lambda_n$, $n \in \omega$, there is a sequence $\{L_{\lambda,k}\}$ of zero-sets in Y such that $V_\lambda = \bigcup_{k\in\omega} L_{\lambda,k}$. For each $\alpha \in \Omega$, $n \in \omega$ and $k \in \omega$, we put

$$K_{\alpha,n,k} = \bigcup\{L_{\lambda,k}\colon \lambda \in \Lambda_n \text{ with } F_\alpha \subset U_\lambda\}.$$

By $F_\alpha \neq \emptyset$, note that $\{V_\lambda\colon \lambda \in \Lambda_n \text{ with } F_\alpha \subset U_\lambda\}$ is locally finite in Y. It follows from [10, Lemma 2.3] that each $K_{\alpha,n,k}$ is a zero-set in Y. Since X is perfectly normal, each F_α is also a zero-set in X. Let

$$\mathcal{E} = \{F_\alpha \times K_{\alpha,n,k}\colon \alpha \in \Omega \text{ and } n, k \in \omega\}.$$

Each member of $\mathcal{E}$ is a zero rectangle in $X \times Y$. Since X is paracompact, the condition (a) in Structural Lemma I is clearly satisfied. For each $\alpha \in \Omega$ and each $n, k \in \omega$, we put

$$\mathcal{G}_{\alpha,n,k} = \{U_\lambda \times V_\lambda\colon \lambda \in \Lambda_n \text{ with } F_\alpha \subset U_\lambda\}.$$

Then we have $K_{\alpha,n,k} \subset \bigcup\{V_\lambda\colon \lambda \in \Lambda_n \text{ with } F_\alpha \subset U_\lambda\} = \bigcup\{G''\colon G \in \mathcal{G}_{\alpha,n,k}\}$. As stated above, $\{G''\colon G \in \mathcal{G}_{\alpha,n,k}\}$ is locally finite in Y. Clearly, $G \in \mathcal{G}_{\alpha,n,k}$ implies $F_\alpha \subset G'$. So the condition (b) in Structural Lemma I is satisfied. We show that $\mathcal{E}$ covers $X \times Y$. Pick any $(x, y) \in X \times Y$. Take some $n_0 \in \omega$ and $\mu \in \Lambda_{n_0}$ with $(x, y) \in U_\mu \times V_\mu$. Since $\mathcal{F}$ is a net of X, there is an $F_\delta \in \mathcal{F}$ such that $x \in F_\delta \subset U_\mu$. Choose $k_0 \in \omega$ with $y \in L_{\mu,k_0}$. Then we have $(x, y) \in F_\delta \times L_{\mu,k_0} \subset F_\delta \times K_{\delta,n_0,k_0} \in \mathcal{E}$. Hence $\mathcal{E}$ covers $X \times Y$. Thus it follows from Structural Lemma I that $\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking. $\square$

Theorem 4.1 yields the following extension of Corollary 3.3.

Corollary 4.2. *Let X be a paracompact σ-space. Let $X \times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent*:

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

By the proof of [11, Theorem 4.1] (or [21, Lemma 2.1]), the conditions (c) and (d) in Structural Lemma II are satisfied for a paracompact Σ-space X and a paracompact P-space Y. So, by this lemma, we have

Theorem 4.3. *If X is a paracompact Σ-space and Y is a paracompact P-space, then every open cover of $X \times Y$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

Comparing Theorems 4.1 and 4.3, it is natural to raise the following question.

Question. Can "σ-space" be replaced by "Σ-space" in Theorem 4.1?

5. Products with a factor defined by topological games

Telgársky [18] introduced the topological game $G(\mathbf{DC}, X)$, where $\mathbf{DC}$ denotes the class of all spaces which have a discrete cover consisting of compact sets.

According to [4], a function s from the family of all closed sets in X to itself is called a *winning strategy for player I* in $G(\mathbf{DC}, X)$ if it satisfies

(a) $s(F) \in \mathbf{DC}$ and $s(F) \subset F$ for each closed set F in X,
(b) if $\{F_n\}$ is a decreasing sequence of closed sets in X such that $s(F_n) \cap F_{n+1} = \emptyset$ for each $n \in \omega$, then $\bigcap_{n\in\omega} F_n = \emptyset$.

A space X is said to be ***DC**-like* if there is a winning strategy for player I in $G(\mathbf{DC}, X)$.

Lemma 5.1 *(Terasawa). Let C be a compact Hausdorff space and Y a space. If $\mathcal{O}$ is a normal cover of $C \times Y$, then there are a family $\{\mathcal{U}_\lambda\colon \lambda \in \Lambda\}$ of finite cozero covers of C and a locally finite cozero cover of $\{V_\lambda\colon \lambda \in \Lambda\}$ of Y such that $\{U \times V_\lambda\colon U \in \mathcal{U}_\lambda \text{ and } \lambda \in \Lambda\}$ refines $\mathcal{O}$.*

This is found in [2, Lemma 1] and [8, Theorem 2.5].

Theorem 5.2. *Let X be a paracompact **DC**-like space and Y a space. Let $\mathcal{O}$ be a normal cover of $X \times Y$. Then the following are equivalent.*

(a) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(b) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

Proof. We only have to show (a) implies (d). Let s be a winning strategy for player I in $G(\mathbf{DC}, X)$. We may assume that $\mathcal{O}$ is a σ-locally finite cozero rectangular cover of $X \times Y$.

Now, for each $n \in \omega$, we will construct a collection $\mathcal{G}_n$ of cozero rectangles, two collections $\mathcal{S}_n$ and $\mathcal{Z}_n$ of zero rectangles, a function $\varphi^n_{n-1} : \mathcal{Z}_n \to \mathcal{Z}_{n-1}$ and a sequence $\{H_k(Z)\colon Z \in \mathcal{Z}_n\}$, $k \in \omega$, of swelling of $\mathcal{Z}_n$ consisting of cozero rectangles, satisfying the following conditions:

(1) $\mathcal{G}_n$ is locally finite and σ-discrete in $X \times Y$.
(2) $\mathcal{S}_n$ is a partial shrinking of $\mathcal{G}_n$.
(3) $\mathcal{Z}_n$ is locally finite and σ-discrete in $X \times Y$.
(4) $\{H_0(Z)\colon Z \in \mathcal{Z}_n\}$ is locally finite and σ-discrete in $X \times Y$.
(5) For each $Z \in \mathcal{Z}_n$, $Z \subset \varphi^n_{n-1}(Z)$.
(6) For each $Z \in \mathcal{Z}_n$, $Z = \bigcap_{k\in\omega} H_k(Z)$ and $\overline{H_{k+1}(Z)} \subset H_k(Z) \subset H_k(\varphi^n_{n-1}(Z))$ $(k = 0, 1, \ldots)$.
(7) $(\bigcup_{i\leqslant n} \mathcal{S}_i) \cup \mathcal{Z}_n$ covers $X \times Y$.
(8) Each member of $\mathcal{G}_n$ is contained in some member of $\mathcal{O}$.
(9) $\bigcup \mathcal{G}_n \subset \bigcup\{H_{n-1}(Z)\colon Z \in \mathcal{Z}_{n-1}\}$.
(10) For each $Z \in \mathcal{Z}_n$, $s(\varphi^n_{n-1}(Z)') \cap Z' = \emptyset$.

Assume that we have already constructed the above ones for $i \leqslant n$.

Take a $Z \in \mathcal{Z}_n$ and fix it. By the assumption of s, there is a discrete collection $\{C_\alpha\colon \alpha \in \Omega_Z\}$ of compact sets in Z' (so in X) such that $s(Z') = \bigcup\{C_\alpha\colon \alpha \in \Omega_Z\}$. Since X is paracompact, there are three discrete swellings $\{W_{\alpha,i}\colon \alpha \in \Omega_Z\}$, $i \in 2$, and $\{L_\alpha\colon \alpha \in \Omega_Z\}$ of $\{C_\alpha\colon \alpha \in \Omega_Z\}$ in X, satisfying for each $\alpha \in \Omega_Z$,

(i) $W_{\alpha,0}$ and $W_{\alpha,1}$ are cozero-sets in X,
(ii) L_α is a zero-set in X,
(iii) $C_\alpha \subset W_{\alpha,0} \subset L_\alpha \subset W_{\alpha,1}$.

Take an $\alpha \in \Omega_Z$. Note that

$$\bigl\{O \cap (C_\alpha \times Y)\colon O \in \mathcal{O} \text{ with } O \cap (C_\alpha \times Z'') \neq \emptyset\bigr\} \cup \bigl\{C_\alpha \times (Y \setminus Z'')\bigr\}$$

is a normal cover of $C_\alpha \times Y$ consisting of cozero rectangles. By Lemma 5.1, there are a collection $\{U_{\lambda,i}\colon i \leqslant k_\lambda$ and $\lambda \in \Lambda_\alpha\}$ of cozero-sets in C_α and a locally finite collection $\{V_\lambda\colon \lambda \in \Lambda_\alpha\}$ of cozero-sets in Y such that

(iv) $C_\alpha = \bigcup_{i\leqslant k_\lambda} U_{\lambda,i}$ for each $\lambda \in \Lambda_\alpha$,
(v) $Z'' \subset \bigcup\{V_\lambda\colon \lambda \in \Lambda_\alpha\} \subset H_n(Z)''$,
(vi) each $U_{\lambda,i} \times V_\lambda$ is contained in $O_{\lambda,i} \cap (C_\alpha \times Y)$ for some $O_{\lambda,i} \in \mathcal{O}$.

Moreover, since $\{V_\lambda\colon \lambda \in \Lambda_\alpha\} \cup \{Y \setminus Z''\}$ is a normal cover of Y, we may assume that $\{V_\lambda\colon \lambda \in \Lambda_\alpha\}$ is locally finite and σ-discrete in Y. Since C_α is a compact subset of a Tychonoff space X, C_α is C-embedded in X (for example, see [3, p. 43]). For each $\lambda \in \Lambda_\alpha$ and $i \leqslant k_\lambda$, there is a cozero-set $U^*_{\lambda,i}$ in X such that $U^*_{\lambda,i} \cap C_\alpha = U_{\lambda,i}$ and $U^*_{\lambda,i} \subset O'_{\lambda,i} \cap W_{\alpha,0} \cap H_0(Z)'$. Since X is normal, there is a finite closed cover $\{E_{\lambda,i}\colon i \leqslant k_\lambda\}$ of C_α such that $E_{\lambda,i} \subset U_{\lambda,i}$ for each $i \leqslant k_\lambda$. Moreover, there are two swellings $\{B_{\lambda,i}\colon i \leqslant k_\lambda\}$ and $\{D_{\lambda,i}\colon i \leqslant k_\lambda\}$ of $\{E_{\lambda,i}\colon i \leqslant k_\lambda\}$ in X such that for each $i \leqslant k_\lambda$,

(viii) $B_{\lambda,i}$ is a cozero-set in X,
(ix) $D_{\lambda,i}$ is a zero-set in X,
(x) $E_{\lambda,i} \subset B_{\lambda,i} \subset D_{\lambda,i} \subset U^*_{\lambda,i}$.

Since $\{V_\lambda \cap Z''\colon \lambda \in \Lambda_\alpha\}$ is a locally finite, σ-discrete, cozero cover of the zero-set Z'' in Y, there is a locally finite, σ-discrete, collection $\{F_\lambda\colon \lambda \in \Lambda_\alpha\}$ of zero-sets in Y such that

(xi) $F_\lambda \subset V_\lambda \cap Z''$ for each $\lambda \in \Lambda_\alpha$ and
(xii) $\bigcup\{F_\lambda\colon \lambda \in \Lambda_\alpha\} = Z''$.

For each $\alpha \in \Omega_Z$ and each $\lambda \in \Lambda_\alpha$, let

$$Z_\lambda(\alpha) = \left(\left(L_\alpha \setminus \bigcup_{i\leqslant k_\lambda} B_{\lambda,i}\right) \cap Z'\right) \times F_\lambda \quad \text{and} \quad H_0(Z_{\alpha,\lambda}) = (W_{\alpha,1} \times V_\lambda) \cap H_0(Z)'.$$

Let $Z^* = (Z' \setminus \bigcup\{W_{\alpha,0}\colon \alpha \in \Omega_Z\}) \times Z''$ and $H_0(Z^*) = H_0(Z)$. Moreover, we let

$$\mathcal{Z}_{n+1}(Z)=\{Z_{\lambda,\alpha}\colon\ \lambda\in\Lambda_\alpha \text{ and } \alpha\in\Omega_Z\}\cup\{Z^*\},$$
$$\mathcal{G}_{n+1}(Z)=\{U^*_{\lambda,i}\times V_\lambda\colon\ i\leqslant k_\lambda,\ \lambda\in\Lambda_\alpha \text{ and } \alpha\in\Omega_Z\} \quad\text{and}$$
$$\mathcal{S}_{n+1}(Z)=\{D_{\lambda,i}\times F_\lambda\colon\ i\leqslant k_\lambda,\ \lambda\in\Lambda_\alpha \text{ and } \alpha\in\Omega_Z\}.$$

Here, letting Z run over $\mathcal{Z}_n$, we put

$$\mathcal{Z}_{n+1}=\bigcup\bigl\{\mathcal{Z}_{n+1}(Z)\colon\ Z\in\mathcal{Z}_n\bigr\},\quad \mathcal{G}_{n+1}=\bigcup\bigl\{\mathcal{G}_{n+1}(Z)\colon\ Z\in\mathcal{Z}_n\bigr\}\quad\text{and}$$
$$\mathcal{S}_{n+1}=\bigcup\bigl\{\mathcal{S}_{n+1}(Z)\colon\ Z\in\mathcal{Z}_n\bigr\}.$$

The function $\varphi^{n+1}_n:\mathcal{Z}_{n+1}\to\mathcal{Z}_n$ is defined by $\varphi^{n+1}_n(\mathcal{Z}_{n+1}(Z))=\{Z\}$ for each $Z\in\mathcal{Z}_n$. Then each member of $\mathcal{G}_{n+1}$ is a cozero rectangle and each member of $\mathcal{Z}_{n+1}$ and $\mathcal{S}_{n+1}$ is a zero rectangle in $X\times Y$. Take any $\widehat{Z}\in\mathcal{Z}_{n+1}$ with $\varphi^{n+1}_n(\widehat{Z})=Z$, that is, $\widehat{Z}\in\mathcal{Z}_{n+1}(Z)$. There is a sequence $\{H_k(\widehat{Z})\}$ of cozero rectangles such that $\widehat{Z}=\bigcup_{k\in\omega}H_k(\widehat{Z})$ and $\overline{H_{k+1}(\widehat{Z})}\subset H_k(\widehat{Z})\subset H_k(Z)$ for each $k\in\omega$. Thus, we have constructed the desired $\mathcal{G}_{n+1}$, $\mathcal{S}_{n+1}$, $\mathcal{Z}_{n+1}$, φ^{n+1}_n and $\{H_k(\widehat{Z})\colon\ \widehat{Z}\in\mathcal{Z}_{n+1}$ and $k\in\omega\}$. It is easily verified that the conditions (1)–(10) are satisfied.

Claim 1. $\bigcap_{n\in\omega}(\bigcup\mathcal{Z}_n)=\emptyset$.

The proof is obtained by the similar way to the combination of the proofs of Claims 1 and 2 in the proof of [19, Theorem 2.1].

Let $H_{n,k}=\bigcup\{H_k(Z)\colon\ Z\in\mathcal{Z}_n\}$ for each $n,k\in\omega$.

Claim 2. $\bigcup\mathcal{Z}_n=\bigcap_{k\in\omega}H_{n,k}=\bigcap_{k\in\omega}\overline{H}_{n,k}$ *for each* $n\in\omega$.

The proof is the same as that of Claim 2 in the proof of [21, Lemma 3.2].

Let $\mathcal{G}=\bigcup_{n\in\omega}\mathcal{G}_n$ and $\mathcal{S}=\bigcup_{n\in\omega}\mathcal{S}_n$. It follows from Claim 1 and (7) that $\mathcal{S}$ is a cover of $X\times Y$. So, by (1), $\mathcal{G}$ is a σ-discrete cover of $X\times Y$. By (2), $\mathcal{S}$ is a zero rectangular shrinking of $\mathcal{G}$. By (8), $\mathcal{G}$ is a cozero rectangular refinement of $\mathcal{O}$. Moreover, in the same way as the last part of the proof of [21, Lemma 3.2], we can verify that $\mathcal{G}$ is locally finite in $X\times Y$. Hence $\mathcal{G}$ and $\mathcal{S}$ are the desired ones. □

If a Hausdorff space X is subparacompact and C-scattered or it has a σ-closure-preserving cover consisting of compact sets, then player I has a winning strategy for $G(\mathbf{DC},X)$, that is, X is **DC**-like (see [18, Theorems 9.7 and 14.7]). So the following is an immediate consequences of Theorem 5.2.

Corollary 5.3. *Suppose that a paracompact space X is C-scattered or has a σ-closure-preserving cover consisting of compact sets and Y is a space. Let $X\times Y$ be a rectangular product and $\mathcal{O}$ an open cover of $X\times Y$. Then the following are equivalent*:

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero rectangular refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero rectangular refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero rectangular refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero rectangular refinement which has a zero rectangular shrinking.*

6. Base-paracompactness of rectangular products

A Hausdorff space X is said to be *base-paracompact* [15] if there is a base $\mathcal{B}$ of X such that $|\mathcal{B}|=w(X)$ and every open cover of X has a locally finite refinement consisting of members of $\mathcal{B}$.

Proposition 6.1. *Let X and Y be base-paracompact spaces. Assume that every normal cover of $X\times Y$ has a locally finite cozero rectangular refinement which has a zero rectangular shrinking. If $X\times Y$ is paracompact, then it is base-paracompact.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be bases of X and Y, respectively, which witness their base-paracompactness. Let $\mathcal{A} \times \mathcal{B} = \{A \times B\colon A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$. Then $\mathcal{A} \times \mathcal{B}$ is a base of $X \times Y$ with $|\mathcal{A} \times \mathcal{B}| = w(X) \cdot w(Y) = w(X \times Y)$.

Let $\mathcal{O}$ be an open cover of $X \times Y$. Since $X \times Y$ is paracompact, $\mathcal{O}$ is normal. By the assumption, $\mathcal{O}$ has a locally finite cozero rectangular refinement $\{U_\lambda \times V_\lambda\colon \lambda \in \Lambda\}$ and its zero rectangular shrinking $\{C_\lambda \times D_\lambda\colon \lambda \in \Lambda\}$. Then there are locally finite subcollections $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$ of $\mathcal{A}$ and $\mathcal{B}$, respectively, such that $C_\lambda \subset \bigcup \mathcal{A}_\lambda \subset U_\lambda$ and $D_\lambda \subset \bigcup \mathcal{B}_\lambda \subset V_\lambda$ for each $\lambda \in \Lambda$. Then it is easily seen that the subcollection

$$\{A \times B\colon A \in \mathcal{A}_\lambda,\ B \in \mathcal{B}_\lambda \text{ and } \lambda \in \Lambda\}$$

of $\mathcal{A} \times \mathcal{B}$ is a locally finite refinement of $\mathcal{O}$. So $\mathcal{A} \times \mathcal{B}$ witnesses the base-paracompactness of $X \times Y$. $\square$

Theorem 3.2 and Proposition 6.1 immediately yield

Corollary 6.2. *Let X and Y be base-paracompact spaces. If $X \times Y$ is paracompact and X-rectangular, then it is base-paracompact.*

Theorem 4.1 and Proposition 6.1 immediately yield

Corollary 6.3. *Let X be a base-paracompact σ-space and Y a base-paracompact space. If $X \times Y$ is paracompact and rectangular, then it is base-paracompact.*

Zhong [22] actually proved that the product $X \times Y$ of a stratifiable space X and a paracompact space Y is rectangular if it is (countably) paracompact. So Corollary 6.3 is an extension of [21, Corollary 4.4].

References

[1] R.A. Alo, H.L. Shapiro, Normal Topological Spaces, Cambridge University Press, London, 1974.
[2] T. Chiba, K. Chiba, Q-paracompactness and closed mappings, Sci. Rep. Tokyo Kyouiku Daigaku Sect. A 11 (1972) 159–162.
[3] L. Gillman, M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, NJ, 1960.
[4] F. Galvin, R. Telgársky, Stationary strategies in topological games, Topology Appl. 22 (1986) 51–69.
[5] T. Hoshina, Extensions of mappings II, in: K. Morita, J. Nagata (Eds.), Topics in General Topology, North-Holland, Amsterdam, 1989, pp. 41–80.
[6] T. Hoshina, K. Morita, On rectangular products of topological spaces, Topology Appl. 11 (1980) 47–57.
[7] K. Morita, Paracompactness and product spaces, Fund. Math. 50 (1962) 223–236.
[8] K. Morita, Čech cohomology and covering dimension for topological spaces, Fund. Math. 87 (1975) 31–52.
[9] K. Morita, Dimension of general topological spaces, in: G.M. Reed (Ed.), Surveys in General Topology, Academic Press, New York, 1980, pp. 297–336.
[10] K. Morita, T. Hoshina, P-embedding and product spaces, Fund. Math. 93 (1976) 71–86.
[11] K. Nagami, Σ-spaces, Fund. Math. 65 (1969) 169–192.
[12] H. Ohta, Topologically complete spaces and perfect maps, Tsukuba J. Math. 1 (1977) 77–89.
[13] H. Ohta, Extensions of zero-sets in the product of topological spaces, Topology Appl. 35 (1990) 21–39.
[14] B.A. Pasynkov, On the dimension of rectangular products, Soviet Math. Dokl. 16 (1975) 344–347.
[15] J.E. Porter, Base-paracompact spaces, Topology Appl. 128 (2003) 145–156.
[16] J.C. Przymusiński, A solution to a problem of E. Michael, Pacific J. Math. 114 (1984) 235–242.
[17] K. Tamano, A note on E. Michael's example and rectangular products, J. Math. Soc. Japan 34 (1982) 187–190.
[18] R. Telgársky, Spaces defined by topological games, Fund. Math. 88 (1975) 193–223.
[19] Y. Yajima, Topological games and products I, Fund. Math. 113 (1981) 141–153.
[20] Y. Yajima, Special refinements and their applications on products, Topology Appl. 104 (2000) 293–308.
[21] Y. Yajima, A base property in paracompact products and its applications, Topology Appl. 142 (2004) 19–30.
[22] N. Zhong, Products with an M_3-factor, Topology Appl. 45 (1992) 131–144.